UNFOLDING SOCIAL HIERARCHIES IN LARGE POPULATION GAMES*

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A B S T R A C T

Consider a large (continuum) population of finitely-lived agents organized in hierarchical levels. Every period, agents are matched to play a certain symmetric game. On the basis of the payoffs obtained, a certain ρ-fraction of those who performed best at each level are promoted upwards. On the other hand, newcomers replacing those who die every period enter at the lowest level and imitate unbiasedly (but subject to noise) the actions adopted at the highest one.

In this context, the (unique) long-run behavior of the system is fully characterized for the whole class of 2×2 coordination games. The results crucially depend on the "institutional" parameter ρ (which reflects how hierarchical – or selective – the society is) and on a purely *ordinal* criterion on the payoffs of the game. In particular, efficient (or inefficient behavior) may prevail in the long run – even when risk-dominated – if promotion in society is (or, respectively, is not) selective enough.

Keywords: Social hierarchies, large population games.

1 Introduction

Social learning in population games has been an important topic of research in recent years. A significant part of this li terature has pursued an evolutionary approach,[1] agents being assumed only boundedly rational when adjusting their behavior over time on the basis of relatively simple rules. These models display two important features:

1. They assume (at least implicitly) that agents have access to relevant payoff information which is used to guide their behavioral adjustments. For example, it is typically supposed that agents can either compute a best response to the current situation, or they can imitate those actions with highest payoff performance.

2. They formulate stylized theoretical frameworks which either display little "institutional" structure (e.g. a globally uniform matching scenario)[2] or have their potential institutional richness masked by a reduced-form specification and some abstract requirement of "monotonicity" (e.g. the Replicator Dynamics and some of its generalizations).

In the present paper, I propose a significant departure from traditional evolutionary analysis in each of two former respects.

First, in contrast with (1), it is assumed that the payoff achieved by the different strategies is not directly observed. Instead, only some indication of their relative performance is obtained through information regarding who has proven to be more successful in rising through the social hierarchy (see below). The motivation here is that, in many cases of interest, it is realistic to assume that the payoff achieved by some strategies can only be observed indirectly. Indeed, I would like to argue that the more complex and multi-faceted a "strategy" is, the harder it usually becomes to attain a clear-cut assessment of its benefits. (For example, think of how difficult it is to evaluate the life-time benefits of a certain choice of education *directly*, i.e. through a clear observation of its "payoff".)

[1]A very partial list includes Foster & Young (1990), Fudenberg and Harris (1992), Kandori, Mailath & Rob (1993), Young (1993), and Samuelson (1994).

[2]There are some important exceptions where the (fixed) interaction pattern is assumed to be local (e.g. Ellison (1993)) or even flexible and endogenously determined through the agents' own adjustment of their, say, location decisions (e.g. Mailath, Samuelson and Shaked (1995), and Ely (1995)). Unlike these papers, our focus here is on a hierarchical segmentation in the interaction pattern, the "re-location" experienced by agents being imposed on them by the mechanism of social promotion. See below for details.

Regarding (2) above, the major novelty displayed by the present framework is embodied by the assumption that individuals are organized and interact within an *evolving* social hierarchy . Specifically, it is postulated that agents, finitely lived, enter the population through level zero and select a certain strategy to be played for the rest of their life. In every period, all those agents who are still alive within each level become internally matched to play a certain bilateral game. Then, on the basis of the payoffs earned, a given fraction $\rho \in (0, 1)$ among those who performed best at each level are "promoted" upwards to the level above.

The paper's main theoretical concern dwells on the question of whether the kind of payoff-responsive institutions described (as parametrized by ρ) may prove effective in compensating for the assumed lack of payoff observability and lead the population towards efficient social behavior. Of course, a key point in this respect pertains to how incoming agents are assumed to choose their life-long strategies. The following "cultural" (status-sensitive) formulation is adopted: among newcomers, the fraction of those choosing each strategy equals their respective frequency in the upper-most echelon of society.

Combining the different considerations explained, we are led to a multi-faceted process of social learning that embodies inter-level imitation, intra-level interaction, and payoff-responsive promotion. These are the core components of the dynamic model studied in this paper. However, for both conceptual and technical reasons (e.g. to ensure the ergodicity of the process), it will also be assumed that both imitation and promotion are perturbed occasionally (i.e. with very small probability) in a stochastic and time-independent manner.

The model outlined has been inspired by some recent work of Harrington (1995). Three crucial differences with it are as follows.

First, Harrington contemplates a model with *no* strategic interaction: the payoff achieved by each agent only depends on her own action. Therefore, in contrast with our emphasis here, his focus is not on games or issues of equilibrium selection.

Second, Harrington postulates that promotion to upper levels is the outcome of a pairwise contest, agents at each level being matched in pairs to this effect. (Thus, in particular, always half of the population at each level is promoted.)

Finally, the framework studied by Harrington does not allow for stochastic noise to play any role in the motion of the system. Therefore, one cannot not obtain the ergodicity conclusions (i.e. independence of initial conditions) which are the essence of our equilibrium-selection results.

Somewhat more distantly, the approach pursued in this paper is also reminiscent of the recent literature that has fruitfully introduced considerations of social status into processes of accumulation and growth. The common starting point of the different papers in this line of research is the *explicit* postulate that agents display a certain (monotone) preference for status. However, the specific mechanism by which agents are supposed to attain status in each particular model is often quite different. For example, in Cole, Mailath & Postlewaite (1992),[3] status is the outcome of a social contest for "mates" that is resolved on the basis of relative wealth. In contrast, Fershtman, Murphy & Weiss (1996) postulate that the status associated to a certain action (in their case, the choice of occupation) is given by the average level of human capital prevailing among those who adopt this action. Naturally, when these diverse considerations are embedded in an intertemporal context, the way in which they affect agents' decisions is also quite diverse, e.g. they reinforce capital accumulation in Cole *et al.*(1992), or may introduce potentially harmful distortions that are detrimental to growth in Fershtman *et al.* (1996).

In the present framework, if the current "status" of an agent is identified with her position in the hierarchy, the role it plays in the dynamics of the model is quite different from that of the above mentioned literature. Here, status only acts as a signalling (or socialization) device that directs the adoption decisions of newcomers. As explained in more detail below, it may be natural to suppose that such notion of status has a tangible effect on payoffs (e.g. it could be postulated that the "game" played at higher levels is unambiguously better). However, even though this would be an appealing motivation for the imitation rule formulated for newcomers, it is not at all a required feature of the model.

I end this Introduction with a summary of the results and a brief discussion of how they compare with received evolutionary literature. As in much of this literature, the focus here is on symmetric 2×2-coordination games, i.e. games with two pure-strategy symmetric equilibria.[4] In this context, it is first shown that a unique outcome is obtained in the long run as the

[3] See also the later papers of Cole, Mailath, and Postlewaite (1996) and Corneo and Jeanne (1996).

[4] In a companion paper (Vega-Redondo (1997)), I analyze the implications of the present approach for the complementary family of 2×2-games which reflect a problem of "asymmetric coordination". In these cases (e.g. the familiar Hawk-Dove game), Nash equilibrium behavior either embodies a fundamental asymmetry or, if symmetric, induces a certain "balance" between the two strategies.

As it turns out, even then one obtains a unique long-run prediction for the model that depends on interesting ways on its underlying parameters. However, in contrast with the present case, the nexus between efficiency and ρ may be quite complex, generally not leading to equilibrium behavior of the underlying game.

stochastic noise becomes arbitrarily small in a suitable sense (i.e. a unique configuration is observed almost surely along any sample path). Moreover, such unique long-run outcome can be identified with one of the equilibria of the underlying game, every player in it choosing a common (equilibrium) strategy. Whether or not this strategy is the efficient one depends on the following two key features of the environment: the selection rate ρ (recall above) and the off-equilibrium payoffs prevailing in the game.

The simplest case arises when the promotion conditions are quite demanding; specifically, when $\rho < 1/2$. Then, the efficient strategy (i.e. the strategy played in the efficient equilibrium) is always selected in the long run, independently of any other considerations (in particular, irrespectively of what are the off-equilibrium payoffs in the underlying game). However, in the polar case where $\rho > 1/2$ (that is, when the environment is not quite as demanding), which strategy ends up prevailing in the long run crucially depends on off-equilibrium payoffs. More precisely, the efficient strategy continues to be selected in the long run if, and (essentially) only if, its off-equilibrium payoff is larger than that of the inefficient strategy.[5]

These conclusions contrast sharply with those found in previous literature in at least two important respects:

(a) First and foremost, the issue of whether efficiency is achieved in the long run hinges upon certain "cultural" or "institutional" features of how society is organized (in particular, the stringency of its promotion requirements).

(b) Second, when these institutional features are such that they allow for the *possibility* of long-run *in*efficiency, the relevant criterion on payoffs which leads to its materialization is both qualitative in nature (i.e. ordinal) and significantly *weaker* than the customary notion of risk dominance.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 carries out the analysis. Section 4 includes the verbal discussion of the results and suggests some extensions. Finally, the formal proofs are contained in the Appendix.

2 The model

Consider a large population of agents, with the cardinality of the continuum, distributed in $n+1$ different social strata (or levels). At every $t = 1, 2, \ldots$, those in each level $k = 0, 1, 2, \ldots, n$ are assumed randomly matched in pairs

[5] As explained in Section 4, this has some bearing on the issue of whether or not cooperation may arise in the long run when agents play a certain restricted version of the finitely repeated Prisoner's Dilemma.

to play a bilateral symmetric game with strategies and payoffs as described in the following table:

	A	B
A	a, a	d, c
B	c, d	b, b

Table 1

This game is assumed to be of the coordination type, i.e. $a > c$, $b > d$. Furthermore, for the sake of concreteness, we take $a > b$, i.e. strategy A is the efficient (equilibrium) strategy.

Agents live for n periods. At every t, a continuum of them (of measure one) start their life by choosing a certain strategy. This choice is conceived as a life-long option (e.g. the learning a profession or the adoption a set of "values") which, once made, becomes irreversibly fixed for the rest of an agent's life.

Every agent enters the social hierarchy at level zero. Then, depending of the (relative) payoffs earned throughout her life, she may escalate up along it. Two essential features of the model need to be specified in this respect: (i) How newcomers choose their strategy; (ii) the promotion and interaction mechanisms prevailing at each level. They are addressed in turn.

Concerning (i), it will be postulated that newcomers independently choose each of the two actions, A and B, with a probability equal to their respective frequencies at the highest level of society. This is the stylized formulation typically proposed by socio-biological models of cultural evolution (see, for example, the classical work by Boyd & Richerson (1985)). In their language, it amounts to identifying those agents who occupy the highest position in the social hierarchy as the "role models" shaping social learning.

As suggested in the Introduction, a natural way to motivate this formulation is to suppose that there is social "prestige" (or status) derived from success in ascending along the social hierarchy. An alternative (but complementary) possibility would be to suppose that the game played at the upper levels of society yields uniformly higher payoffs than those which can be earned at lower ones. For example, suppose that payoffs (taken to be non-negative) are multiplied by a certain level-specific factor which is *common* to all strategies but increases in a sufficiently steep fashion with the social level at which the game is played. This would not affect our analysis,[6] but is an appealing "tangible" materialization of what the social hierarchy implies.

[6] Our results only depend on the ranking of the different possible payoffs, which is obviously not affected by such a procedure.

Under these conditions, it is reasonable to suppose that agents will want to mimic the behavior found to be more prevalent at the highest level, with the hope that they themselves will reach this level as well.

As for (ii) above, I shall follow Harrington (1995) in making the simplifying assumption that, at each t, only those agents who have not yet suffered a set-back in their promotion at some earlier time are still "in the race". This is equivalent to the rather realistic notion that promotion remains a possibility only for those agents who are of the "right" (i.e. youngest) age for the level in question. Among these, the contest for promotion is structured as follows. First, agents are randomly matched within each level $k = 0, 1, ..., n-1$ to play the game. Then, given the profile of payoffs resulting from this interaction, only a fraction ρ of those agents who at each level have obtained the highest payoffs is promoted upwards, the remaining fraction $(1-\rho)$ being forced out of the "race" for the rest of their life. Intuitively, the parameter $\rho \in (0,1)$ reflects the stringency of the promotion requirements prevailing in the society and will be a key consideration in the analysis.

Formally, the process described can be conceived as a dynamical system on the state space $\Omega \equiv [0,1]^n$. At any given t, the state of the system is the vector $\mu(t) = (\mu_1(t), \mu_2(t), ..., \mu_n(t))$, specifying the fraction of the population $\mu_k(t)$ adopting strategy A at each level $k \geq 1$. Since newcomers are assumed to adopt their strategy by independent and unbiased imitation of the highest level, we shall invoke the Law of Large Numbers[7] and simply make $\mu_0(t) = \mu_n(t)$, i.e. the fraction $\mu_0(t)$ of A-adopters among newcomers at t exactly equals that prevailing at the highest level.

Similarly, the transitions across levels induced by the promotion mechanism may be formulated in a purely deterministic fashion if, due to the large numbers involved, one rules out any matching-induced uncertainty in the aggregate. Let

$$f : [0,1] \to [0,1], \quad f(x) = x',$$

stand for the function which specifies the fraction of agents x' choosing strategy A at any level $k+1$ when, for $k = 0, 1, ..., n-1$, this fraction was x at level k during the preceding period. The particular form of this function – cf. (7) or (13) below – depends on the particular scenario under consideration (i.e. on the assumptions made on the game payoffs).

[7]We abstract here from the well-known but delicate issues arising in connection with an application of the Law of Large Numbers to a continuum of random variables. (See, for example, Feldman & Gilles (1985) or Judd (1985).)

Building upon the particular $f(\cdot)$ to be applied in each case, the overall dynamics may be formalized as follows:

$$\mu_{k+1}(t+1) = f(\mu_k(t)), \quad k = 1, ..., n-1, \tag{1}$$
$$\mu_1(t+1) = f(\mu_n(t)). \tag{2}$$

On the one hand, (1) reflects the *promotion dynamics* described in (ii) above as it applies to each level $k = 1, 2, .., n-1$. On the other hand, (2) *jointly* embodies the *socialization process* experienced by the generation of newcomers at t (recall (i) above) together with the ensuing *promotion dynamics* by which a fraction of this generation is promoted to level 1. Interpreting indices as "modulo n", (1) and (2) may be compactly expressed as follows

$$\mu_{k+1}(t+1) = f(\mu_k(t)), \quad k = 1, ..., n, \tag{3}$$

with

$$F : \Omega \to \Omega, \quad \mu(t+1) = F(\mu(t)),$$

representing the corresponding (vector-valued) function.

The function $F(\cdot)$ always displays a multiplicity of fixed (or rest) points; for example, $(1, 1, ..., 1)$ and $(0, 0, ..., 0)$ are two trivial such instances under any specification of the parameters. This implies, in particular, that the limit behavior of the system (in particular, its long-run predictions) cannot be dissociated from initial conditions.

To address such "history dependence", it is a standard approach to introduce small perturbations into the system with the hope that this may help discriminate among alternative long-run outcomes on the basis of their differential robustness. A wide variety of alternative specifications could be considered here with identical implications.[8] For concreteness, it will be simply assumed that, for each $k = 1, 2, ..., n$, there is some common and "small" probability $\varepsilon > 0$ (independent across time and levels) such that each of the equations in (3) is perturbed as follows:

$$\mu_{k+1}(t+1) = f(\mu_k(t)) + \eta_k(t), \tag{4}$$

where $\eta_k(t)$ is a random variable, again independent across time and levels, with support on $[-f(\mu_k(t)), 1 - f(\mu_k(t))]$ – i.e. $f(\mu_k(t)) + \eta_k(t)$ has support

[8] For example, one could allow the probabilities with which perturbations occur at different levels to be different – even infinitesimal of different orders. As an extreme instance of this, some of these probabilities (albeit not all) could be made identically zero. Another possible variation of the model would have the stochastic perturbation operate solely on the socialization dynamics, never on the promotion mechanism. It can be checked that none of the previous alternative formulations would alter the essential conclusions of the analysis.

on $[0, 1]$ – and a continuous density $\phi_\lambda(\cdot \mid \mu_k(t))$. The value of $\lambda \in \mathbb{R}_{++}$ will be used to parametrize the "shape" of the noise, as explained below.

Thus, with some relatively large (independent) probability $1 - \varepsilon$, the promotion dynamics from level k to $k+1$ is taken to proceed unperturbed, as described by (3). On the other hand, with the residual probability ε, this dynamics is perturbed by a certain aggregate shock that, with positive density, leads to any ensuing profile at level $k+1$.

The perturbations contemplated in (4) are similar in spirit to those typically postulated by modern Evolutionary Game Theory. They are to be interpreted as rather infrequent disruptions of the core dynamics of the process, reflecting those unmodelled factors (e.g. migration or some other "external shocks") that may occasionally interfere with it. In contrast with the approach often pursued in finite-population scenarios (cf. Kandori, Mailath and Rob (1993) or Young (1993)) this stochastic noise is *not* formulated as the outcome of some individual-based independent mutation, but directly as an aggregate perturbation. This is motivated by the fact that, in the continuum-population framework considered here, stochastic independent noise at the individual level poses fundamental problems, both of a conceptual and technical nature.[9]

The formulation contemplated in (4) is more akin to that postulated by Foster and Young (1990) and Fudenberg and Harris (1992). However, there is the important difference that, in their framework, *some* aggregate perturbation (however small) is supposed to occur a.s. every period. In contrast, the present approach postulates that the aggregate perturbation takes place (in every period and at every level) only with some probability ε . In this sense, it responds to a rather different conceptual interpretation of the aggregate noise, which is best conceived here as triggered by specific events (or shocks) that occur only rather infrequently.

As advanced, it seems natural to insist that the perturbations considered be only small. In our context, this idea may be approached in two complementary ways.

- First, one may require that the probability ε with which an aggregate perturbation occurs at any level should be small. Formally, this will be captured by making $\varepsilon \to 0$ (see (5)).

- Secondly, one may also demand that, *given* that a perturbation occurs, smaller "shifts" are substantially more likely than larger ones. Only

[9] Recall Footnote 7 on the technical problems arising when considering a continuum of independent random variables. But even if these problems could be tackled and, say, the Law of Large Numbers could be suitably invoked, this would do away with the stochastic nature of the system and preclude its desired ergodicity.

some of our results will build upon this idea. To make it precise, we shall find it useful to rely on a λ-parametrization of the perturbation density that satisfies:

A.1 $\forall \vartheta > 0 \; \exists \bar{\lambda} > 0$ such that if $\lambda \leq \bar{\lambda}$, the following applies:
$\forall \mu_n \in \{0, 1\}$,[10] $\forall a, b \in [-\mu_n, 1 - \mu_n]$,

$$|a| > |b| \Rightarrow \frac{\int_{1 \geq |\eta_k| \geq |a|} \phi_\lambda(\eta_k \mid \mu_k) \, \mathrm{d}\eta_k}{\int_{1 \geq |\eta_k| \geq |b|} \phi_\lambda(\eta_k \mid \mu_k) \, \mathrm{d}\eta_k} \leq \vartheta.$$

The above condition simply expresses the notion that a decreasing value for λ parametrizes conditions where perturbations of larger size become progressively less likely. In the limit, the relative ex-ante likelihood of any arbitrarily small increase in the perturbation falls to zero as λ becomes small. In a heuristic sense, such a small value for λ can be viewed as the large-population counterpart of the vanishing small mutation rate often contemplated by finite-population evolutionary models (recall above).[11]

Further pursuing the analogy with customary evolutionary models, a natural condition to demand from the perturbation is that it display some natural symmetry (i.e. upward and lower shifts display similar probabilities). Restricted again to the two extreme profiles $\mu_n = 0, 1$,[12] this is the interpretation of the following final assumption:

A.2 $\forall \lambda > 0$, $\forall \eta \in [0, 1]$, $\phi_\lambda(\eta \mid 0) = \phi_\lambda(-\eta \mid 1)$.

3 Analysis

Fix λ and for any $\varepsilon > 0$ consider the Markov process on Ω induced by $F(\cdot)$ and the perturbation contemplated in (4). Denote this Markov process by P_ε with transition probabilities $P_\varepsilon(\cdot \mid \mu) \in \Delta(\Omega)$ for each $\mu \in \Omega$, where $\Delta(\Omega)$ stands for the set of Borel probability measures on Ω endowed with the topology of weak convergence. A standard result (see e.g. Meyn and Tweedie (1993)) guarantees that the process has a unique limit distribution which summarizes the long-run behavior of the process. That is:

Proposition 1 *For any $\varepsilon > 0$, the process P_ε converges to a unique invariant distribution $\zeta_\varepsilon \in \Delta(\Omega)$, independently of initial conditions.*

[10] In principle, this condition could be demanded for any value of μ_n. However, note that it is just contemplated for the two extreme profiles $\mu_n = 0, 1$.

[11] Of course, this heuristic analogy is far from accurate, as explained in Footnote 7.

[12] The considerations explained in Footnote 10 apply here as well.

Our concern will be to study the long-run behavior of the system (as summarized by ζ_ε) when $\varepsilon \to 0$. More specifically, the focus will be on the limit invariant distribution

$$\zeta^* \equiv \lim_{\varepsilon \to 0} \zeta_\varepsilon \tag{5}$$

obtained when the permutation probability becomes infinitesimal. The argument involved in characterizing ζ^* – showing, in particular, that it is a well-defined element of $\Delta(\Omega)$ – proceeds in the following two steps.

First, it must be shown that the infinite iteration of the unperturbed dynamics, F^∞, converges to limit behavior displaying suitable long-run regularities. This will be the content of Propositions 4 and 7, respectively concerning each of the two scenarios under consideration. For our present purposes, the key implication here is that, for each $\mu \in \Omega$, the sequence

$$\left\{ \frac{1}{m+1} \sum_{r=0}^{m} \delta_{F^r(\mu)} \right\}_{m=0}^{\infty} \tag{6}$$

converges weakly to some well-defined probability measure $R(\cdot \mid \mu) \in \Delta(\Omega)$, where $\delta_{F^r(\mu)}$ stands for the measure fully concentrated in $F^r(\mu)$. To establish this conclusion is indeed the first "logical step" in the characterization of ζ^*. However, for expositional reasons, it is useful to postpone its detailed statement and proof to the point below where each of the two alternative scenarios is studied in detail (cf. Subsections 3.1 and 3.2).

Assuming, therefore, that the sequence in (6) converges weakly to some probability measure in $\Delta(\Omega)$, the second step in the characterization of ζ^* proceeds as follows. For any given $k = 1, 2, ..., n$, denote by Q^k the Markov process on Ω whose transition probability measure $Q^k(\cdot \mid \mu)$ embodies the following two components:

- the *perturbed* promotion dynamics given by (4) operating on the transition from level k to level $k+1$, together with
- the *unperturbed* promotion dynamics (3) applied on the transitions from all other $k' \neq k$.

Formally, Q^k may be defined through the marginals $Q_i^k(\cdot \mid \mu)$ on each component of the state space as follows:

$$\begin{aligned} Q_{k'}^k(\cdot \mid \mu) &= \delta_{f(\mu_{k'-1})} \quad (k' \neq k+1) \\ Q_{k+1}^k(E \mid \mu) &= \int_{E - f(\mu_k)} \phi_\lambda(y \mid \mu_k)\, dy \end{aligned}$$

where $\delta_{f(\mu_{k'-1})}$ represents the probability measure concentrated in $f(\mu_{k'-1})$ and $E - f(\mu_k) \equiv \{y : f(\mu_k) + y \in E\}$ for any Borel subset E of $[0, 1]$.

Next, on the basis of the transition probabilities $\{Q^k(\cdot \mid \mu)\}_{k=1}^n \in \Delta(\Omega)$ just defined, let Q denote the Markov process on Ω obtained when the promotion dynamics captured by $F(\cdot)$ is perturbed only at *one* level k, each of these chosen with equal *ex ante* probability. Formally,

$$Q(B \mid \mu) = \frac{1}{n} \sum_{k=1}^{n} Q^k(B \mid \mu)$$

for any Borel subset B of Ω.

Finally, let $QR(\cdot \mid \mu)$ stand for the probability measure on Ω obtained by composing R to Q from state μ, i.e. first applying the transitions prescribed by Q, then by R (see the Appendix for the precise formal details). On the basis of such composition, the following result provides a very useful characterization of ζ^*.

Proposition 2 *For any fixed $\lambda > 0$, the sequence of invariant distributions $\{\zeta_\varepsilon\}$ converges weakly to some $\zeta^* \in \Delta(\Omega)$ as $\varepsilon \to 0$. Moreover, ζ^* is the unique invariant measure of the Markov process QR.*

Proof. See the Appendix.

The previous Proposition follows from a general result established by Karandikar, Mookherjee, Ray & Vega-Redondo (1996) for general perturbed Markov processes (see Theorem A in the Appendix). It leads to the following rather intuitive way of computing the limit invariant distribution ζ^*:

(i) First, identify the limit states of the unperturbed process as the only possible candidate states to lie in the support of ζ^*.

(ii) Second, evaluate the relative likelihood of transitions across these states by means of a *single* perturbation at some level *and* the ensuing operation of the unperturbed process.

Two different scenarios which (generically)[13] partition the family of 2×2 *coordination* games need to be distinguished in the analysis. Their distinction only depends on a comparison of the off-equilibrium payoffs, c and d in Table 1.

[13]For non-generic payoff configurations (e.g. when $c = d$), the analysis turns out to be quite analogous to the generic cases studied here. For $\rho < 1/2$, one obtains exactly the same conclusions, i.e. long-run selection of the efficient strategy. Instead, when $\rho > 1/2$, both strategies display identical weight in the limit distribution.

The first scenario, labelled Scenario I, is characterized by the inequality $d > c$. In this case, the efficient strategy A yields a higher payoff than the other one B for both on- and off-equilibrium plays. It implies, in particular, that the efficient strategy is risk-dominant (in the sense of Harsanyi & Selten (1988)), but is substantially stronger.

The second scenario, Scenario II, displays the opposite inequality $c > d$. In this case, the inefficient strategy leads to a higher off-equilibrium payoff than that obtained by the efficient strategy. In line with the previous observation, note that this inequality is consistent with the efficient strategy being risk dominant. Moreover, it *always* applies when the inefficient strategy is risk dominant.

Each of these two scenarios is now addressed in turn.

3.1 Scenario I: $d > c$

In view of (6) and Proposition 2, our first task must be to identify the function $f(\cdot)$ formalizing the promotion dynamics at *any* level $k = 0, 1, 2, ..., n-1$ (recall (1) and (2)). In the present scenario, it turns out to be of the following form:

$$f(x) = \begin{cases} \frac{1}{\rho}x^2 & 0 \le x \le \frac{1-\sqrt{2\rho-1}}{2} & \text{(a)} \\ \frac{1}{\rho}(2x - x^2 + \rho - 1) & \frac{1-\sqrt{2\rho-1}}{2} \le x \le \frac{1-\sqrt{4\rho-3}}{2} & \text{(b)} \\ \frac{1}{\rho}x & \frac{1-\sqrt{4\rho-3}}{2} \le x \le \frac{1+\sqrt{4\rho-3}}{2} & \text{(c)} \\ \frac{1}{\rho}(2x - x^2 + \rho - 1) & \frac{1+\sqrt{4\rho-3}}{2} \le x \le \frac{1+\sqrt{2\rho-1}}{2} & \text{(d)} \\ \frac{1}{\rho}x^2 & \frac{1+\sqrt{2\rho-1}}{2} \le x \le \sqrt{\rho} & \text{(e)} \\ 1 & \sqrt{\rho} \le x \le 1. & \text{(f)} \end{cases} \tag{7}$$

For small enough values of ρ, some of the ranges (a)-(f) may be empty or not well defined. In that case, the above specification of $f(\cdot)$ is interpreted in an obvious fashion. (For example, if $\rho < \frac{1}{2}$, (a) and (e) merge into one continuous range, with (b), (c), and (d) becoming unapplicable. On the other hand, if $\rho > \frac{3}{4}$, all six ranges are well-defined and non-empty.)

Addressing the six x-ranges which define $f(\cdot)$ in turn, I now explain in some detail the formulation associated to each of them. Range (a) applies to those values of $x < 1/2$ such that $2(1-x)x \le 1-\rho$ and $x^2 \le \rho$. If x

is within this range, all those individuals who play strategy A and meet B players (there are $(1-x)x$ of them) do not get promoted. For they receive a lower payoff than those players (of A and B type) who meet their own type (there are $x^2+(1-x)^2$ of them) and we have:

$$x^2+(1-x)^2 = 1-2(1-x)x \geq \rho.$$

Thus, everyone promoted must be among those who meet someone of their own type. On the other hand, all A players who meet A players are promoted upwards since they obtain the highest payoff in the population and their total size x^2 does not exceed the promotion quota ρ. Combining these facts, one concludes that the relevant frequency of individuals adopting A at the next upper level in the following period must be $\frac{1}{\rho}x^2$, as indicated in (a) above. (Note that the total number of relevant individuals – i.e. those still "eligible" for promotion – falls at the rate ρ across consecutive levels.)

Range (b) considers those values of $x < 1/2$ such that $2(1-x)x \geq 1-\rho \geq (1-x)x$. As before, the task is to compute the number of A players which are promoted upwards. First, note that

$$2(1-x)x \geq 1-\rho \Rightarrow x^2 \leq \rho.$$

Thus, all A players meeting A players (x^2 of them) are promoted. Suppose that the second inequality defining the range of (b) is satisfied strictly. This implies that $1-\rho > (1-x)x$. Then, among A players who meet B players, not all can be promoted since

$$\rho < 1-(1-x)x.$$

Thus, only the residual left from ρ after the players who meet their own type have been promoted is available for additional promotion. This implies that only the total number $x^2+\rho-(x^2+(1-x)^2)$ of A players will be promoted. Once this number is again scaled by ρ, the expression in (b) obtains.

Consider now range (c). This corresponds to those values of x which satisfy $(1-x)x \geq 1-\rho$. In this case, all those agents who are not promoted are chosen from B players meeting A players, which implies that all A players are promoted (i.e. x of them). This leads to (c) after dividing by ρ.

Ranges (d) and (e) are symmetric to (b) and (a), respectively, with $x > 1/2$ instead of the opposite inequality. Finally, the expression for range (f) follows from the fact that if $x^2 \geq \rho$, then only A players (specifically, those who meet A players) can be promoted. This implies that only this kind of players will be present at the next upper level in the following period.

Our next task involves identifying the fixed points and other useful global properties of the function $f(\cdot)$. This is the purpose of the following Proposition.

Proposition 3 *Let $f(\cdot)$ be defined by (7) and denote $\xi \equiv \min\{\rho, 1-\rho\}$. The set of fixed points of $f(\cdot)$ is $\{0, 1, \xi\}$. Moreover, for all $x \in (0,1)$,*

$$f(x) > x \Leftrightarrow x > \xi. \tag{8}$$

Proof. See the Appendix.

By way of illustration, Figures 1 and 2 below depict the function $f(\cdot)$ in (7) for the two rather extreme values of $\rho = 0.2$, 0.8.

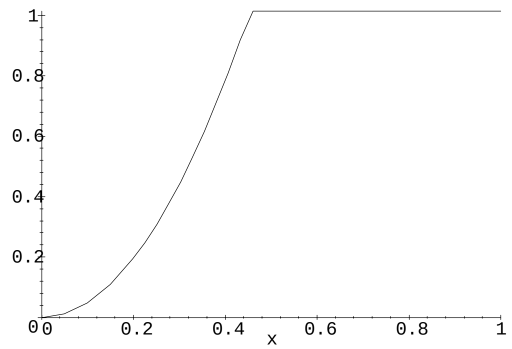

Figure 1: $f(\cdot)$ – Scenario I, $\rho = 0.2$

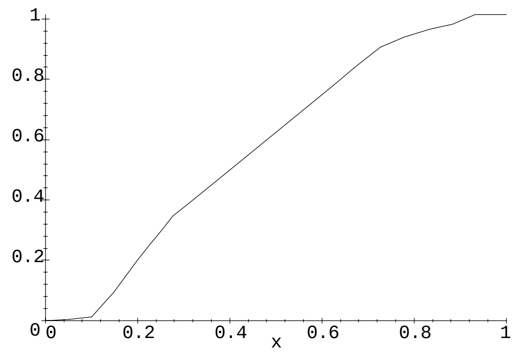

Figure 2: $f(\cdot)$ – Scenario I, $\rho = 0.8$

Next, we characterize the long-run behavior of the (unperturbed) dynamics resulting from $F(\cdot)$ – (uniquely induced from $f(\cdot)$) in the present Scenario I. To do so in Proposition 4 below, the following notation needs to be introduced.

Denote:

$$\Lambda_I = \left\{ \begin{array}{c} (\mu^1, \mu^2, ..., \mu^n) \in \Omega^n : \mu^{j+1} = F(\mu^j), \ \forall j = 1, 2, ..., n-1, \\ \mu^1 = F(\mu^n), \ \mu^1 \in \{0, 1, \xi\}^n \end{array} \right\}. \tag{9}$$

This set represents the collection of n-cycles of $F(\cdot)$ such that, at the beginning of the cycle (and, therefore, throughout it as well by Proposition 3) the frequency of A players at any level k is either 0, 1, or ξ.[14] By construction, these n-cycles define corresponding limit sets for the unperturbed dynamics. The claim that, reciprocally, they induce as well the only limit sets of this dynamics is established by the following result.

Proposition 4 *Assume the payoff conditions of Scenario I. Then, given any initial condition $\mu(0) \in \Omega$, the induced path $\{\mu(t) = F^t(\mu(0))\}_{t=1}^{\infty}$ satisfies:*[15]

$$\lim_{r\to\infty} [\mu(rn+1), \mu(rn+2), ..., \mu(rn+n)] \in \Lambda_I.$$

Proof. See the Appendix.

As a direct corollary of the argument used in the Appendix to establish the previous result, it follows that only those n-cycles in Λ_I which involve "homogeneous" generations (i.e. with a profile $x = 0, 1$) qualify as locally stable for the unperturbed dynamics. Formally, denote:

$$\tilde{\Lambda}_I = \left\{ \begin{array}{c} (\mu^1, \mu^2, ..., \mu^n) \in \Omega^n : \mu^{j+1} = F(\mu^j),\ \forall j = 1, 2, ..., n-1, \\ \mu^1 = F(\mu^n),\ \mu^1 \in \{0,1\}^n \end{array} \right\}. \quad (10)$$

Corollary 1 *Assume the payoff conditions of Scenario I. Then, the set of n-cycles which are locally stable*[16] *for the dynamics induced by $F(\cdot)$ is given by the set $\tilde{\Lambda}_I$.*

From the preceding Corollary, the requirement of local stability imposed on the unperturbed system already narrows down substantially the possible long-run (robust) predictions of the model. To proceed in this direction even further, we now resort to a more stringent requirement of *stochastic stability.*

Denote by $\mathbb{1}$ the state $(1, 1, ..., 1)$ where the efficient strategy A is played at each level. Building upon the characterization provided by Proposition 4, our next result establishes that, within Scenario I, only the constant n-cycle $(\mathbb{1}, \mathbb{1}, ..., \mathbb{1})$ is (generically) robust to the consideration of infrequent and small noise.

[14] Note that there is no requirement of minimality on the n-cycle. Thus, in particular, a fixed point (i.e. a 1-cycle) gives rise to a constant n-cycle.

[15] Throughout the paper, we adopt the standard notational convention that the superindex displayed by a *function* reflects the number of the iterations that it is applied. This should create no confusion with superindices used to identify different particularizations of a certain vector-valued variable, as in the definition of the set Λ_I in (9).

[16] Here, the concept of a locally stable cycle is the usual one. Informally, it merely involves the requirement that, for *each* state in the cycle, there is some sufficiently small neighborhood of it such that all trajectories of the system which start in any one of these neighborhoods converge to the cycle.

Theorem 1 *Consider Scenario I, assume A.1 and A.2, and let* $\rho \neq 1/2$.[17] $\forall \delta \in (0,1)\ \exists \bar{\lambda} > 0$ *such that* $\lambda \leq \bar{\lambda} \Rightarrow \zeta^*(\mathbb{1}) \geq 1 - \delta$.

Proof. See the Appendix.

As explained above, a small value for λ may be viewed as a situation where any perturbation which occurs is very likely to be of quite small *magnitude*. In this light, Theorem 1 may be interpreted to assert that if

(a) the system is very infrequently perturbed (i.e. ε is infinitesimal) and,
(b) when a perturbation does occur, its magnitude is small (as given by λ),

then the long-run behavior of the system is largely concentrated around the state $\mathbb{1}$ where every agent at each level plays the efficient strategy A. This result holds generically in the parameter ρ; specifically, for *all* environments where the selection rate is not *exactly* equal to $1/2$.

As stated by the above result, an arbitrarily sharp selection outcome (i.e. a choice of δ close to zero) generally depends on λ being sufficiently small. This raises the natural question of whether less drastic demands on λ may still allow for some interesting (although not as sharp) conclusions. This is the purpose of the next result, which represents a further illustration of the potential of Proposition 2 as a quite useful characterization of ζ^*.

Let

$$\alpha_\lambda \equiv \int_\xi^1 \phi_\lambda(\eta \mid 0)\, \mathrm{d}\eta \tag{11}$$

$$\beta_\lambda \equiv \int_{-1}^{-(1-\xi)} \phi_\lambda(\eta \mid 1)\, \mathrm{d}\eta \tag{12}$$

and note that, under A.2, one has $\alpha_\lambda > \beta_\lambda$ if $\rho \neq 1/2$. Denote $r \equiv 2^n - 1$, where recall that $n+1$ is the number of levels considered in the model.

A.3 Let n and λ satisfy $\frac{1}{n^{2r}} \left(\frac{1-\alpha_\lambda}{1-\beta_\lambda}\right)^{r-n} \frac{\alpha_\lambda}{\beta_\lambda} > 1$.

Proposition 5 *Consider Scenario I and assume A.3. Then,* $\zeta^*(\mathbb{1}) > \zeta^*(\mu)$ *for all* $\mu \in \mathbf{supp}(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.

Of course, if $\rho \neq 1/2$, A.1 and A.2 imply A.3 for small enough λ. In this sense, the above selection result covers a wider scenario than that of Theorem 1, although with a weaker conclusion. On the other hand, it is straightforward to see that A.2 *alone* implies A.3 if $n = 1$ (i.e. when there are just two levels), provided $\rho \neq 1/2$. In view of this latter observation we have:

[17] When ρ is exactly equal to $1/2$, unique long-run selection no longer obtains and all states belonging to $\tilde{\Lambda}_I$ are equally likely long-run states.

Corollary 2 *Consider Scenario I, let $\rho \neq 1/2$, and assume A.2 and $n = 1$. Then, $\zeta^*(\mathbb{1}) > \zeta^*(\mu)$ for all $\mu \in$ **supp**$(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.*

The above results indicate that, under the conditions specified for Scenario I, the efficient state $\mathbb{1}$ arises as the more prevalent configuration of the system in the long run (either arbitrarily so if λ is chosen sufficiently small, or at least relative to other states if the conditions of Proposition 5 or Corollary 2 apply). These conclusions, which hold uniformly for all promotion rates $\rho \in (0,1)$ such that $\rho \neq 1/2$, are to be contrasted with those obtained below when the payoff conditions are those of Scenario II. A combined discussion of both cases and their underlying intuition will be conducted in Section 4.

3.2 Scenario II: $c > d$

As for Scenario I, we start by characterizing the long-run dynamics induced by the unperturbed process. In the present case, the function $f(\cdot)$ formalizing promotion dynamics is of the following form:

$$f(x) = \begin{cases} \frac{1}{\rho}x^2 & 0 \leq x \leq \frac{1-\sqrt{4\rho-3}}{2} & \text{(a')} \\ \frac{1}{\rho}(x+\rho-1) & \frac{1-\sqrt{4\rho-3}}{2} \leq x \leq \frac{1+\sqrt{4\rho-3}}{2} & \text{(b')} \\ \frac{1}{\rho}x^2 & \frac{1+\sqrt{4\rho-3}}{2} \leq x \leq \sqrt{\rho} & \text{(c')} \\ 1 & \sqrt{\rho} \leq x \leq 1. & \text{(d')} \end{cases} \tag{13}$$

As in (7), some of the ranges (a')-(d') above may be empty or not well-defined, in which case there is a corresponding reduction in the relevant cases that need to be considered (e.g. if $\rho < \frac{3}{4}$, range (b') disappears and $f(x) = \frac{1}{\rho}x^2$ for all $x \leq \sqrt{\rho}$.)

I turn to explaining the formulations specified in (7) for each of the four ranges. Range (a') applies to those values of $x < 1/2$ such that $x(1-x) \leq 1-\rho$ and $x^2 \leq \rho$. Within this range for x, the A players who meet B players receive the lowest possible payoff. Therefore, only those A players who meet players of the same type (x^2 of them) are promoted. Scaling this magnitude by the factor ρ at which the relevant population size decreases across consecutive levels, one is led to the expression contemplated in (a').

Next, Range (b') corresponds to those x such that $x(1-x) \geq 1-\rho$. In this case, some of the A players who meet B players are promoted – specifically, the fraction is $x(1-x) - (1-\rho)$. When this fraction is added to x^2 (i.e.

those A players meeting their own type), (b') is obtained after normalizing the result by the factor ρ.

Range (c') is simply the counterpart to (a') if $x > 1/2$ and $x^2 \leq \rho$. Finally, Range (d') considers those x such that $x^2 \geq \rho$. In this case, no B players are promoted (only A players who meet A players may rise to the upper level). This implies that the frequency of A players in the upper level next period must be 1, which is the content of (d').

The analysis undertaken here is quite parallel to that conducted in the preceding subsection for Scenario I. Consequently, the presentation that follows is kept rather terse. First, we identify the fixed points as well as other global properties of the function $f(\cdot)$, as presently formulated in (13).

Proposition 6 *Let the function $f(\cdot)$ be defined by (13). Its set of fixed points is $\{0, 1, \rho\}$. Moreover, for all $x \in (0,1)$,*

$$f(x) > x \Leftrightarrow x > \rho.$$

The preceding conclusions are illustrated below for $\rho = 0.2$, 0.8.

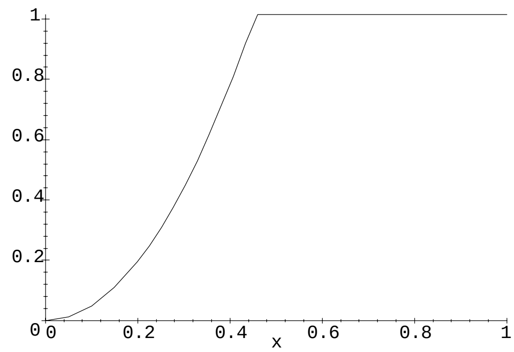

Figure 3: $f(\cdot)$ – Scenario II, $\rho = 0.2$

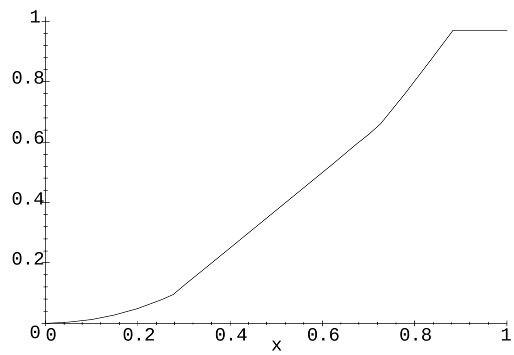

Figure 4: $f(\cdot)$ – Scenario II, $\rho = 0.8$

Next, we characterize the long-run dynamics of the unperturbed process. To this end, denote:

$$\Lambda_{II} = \left\{ \begin{array}{c} (\mu^1, \mu^2, ..., \mu^n) \in \Omega^n : \mu^{j+1} = F(\mu^j), \ \forall j = 1, 2, ..., n-1, \\ \mu^1 = F(\mu^n), \ \mu^1 \in \{0, 1, \rho\}^n \end{array} \right\}$$

where, in contrast with (9), note that the interior value for μ_k appearing here is always ρ, even when it exceeds 1/2. This derives from the fact that, as established by Proposition 6, $x = \rho$ is now the unique interior zero of the function $f(\cdot)$ in every case.

By construction, the n-cycles in Λ_{II} are limit cycles. The reciprocal conclusion is established by the following result.

Proposition 7 *Assume the payoff conditions of Scenario II. Then, given any initial condition $\mu(0) \in \Omega$, the induced path $\{\mu(t) = F^t(\mu(0))\}_{t=1}^{\infty}$ satisfies:*

$$\lim_{r \to \infty} [\mu(rn+1), \mu(rn+2), ..., \mu(rn+n)] \in \Lambda_{II}.$$

Proof. See the Appendix.

As the counterpart of (10), now denote:

$$\tilde{\Lambda}_{II} = \left\{ \begin{array}{c} (\mu^1, \mu^2, ..., \mu^n) \in \Omega^n : \mu^{j+1} = F(\mu^j), \ \forall j = 1, 2, ..., n-1, \\ \mu^1 = F(\mu^n), \ \mu^1 \in \{0, 1\}^n \end{array} \right\}$$

Corollary 3 *Assume the payoff conditions of Scenario II. Then, the set of n-cycles which are locally stable for the dynamics induced by $F(\cdot)$ is given by the set $\tilde{\Lambda}_{II}$.*

The preceding results permit a full characterization of the (robust) limit sets of the unperturbed dynamics in the present scenario. Once the stochastic perturbation is introduced, the above multiplicity drastically shrinks again, as established by the following Theorem. Let $\mathbb{O} \equiv (0, 0, ..., 0)$ denote the state where every player at each level adopts the inefficient strategy B.

Theorem 2 *Consider Scenario II.*[18]
(a) Assume $\rho < 1/2$. Then, $\forall \delta \in (0,1)\ \exists \bar{\lambda} > 0$ s. t. $\lambda \leq \bar{\lambda} \Rightarrow \zeta^*(\mathbb{1}) \geq 1 - \delta$;
(b) Assume $\rho > 1/2$. Then, $\forall \delta \in (0,1)\ \exists \bar{\lambda} > 0$ s. t. $\lambda \leq \bar{\lambda} \Rightarrow \zeta^*(\mathbb{O}) \geq 1 - \delta$.

[18] The counterpart of Footnote 17 applies here as well.

Proof. See the Appendix.

The preceding result points to a sharp effect of the selection rate ρ on the long-run behavior of the system. If promotion is sufficiently selective (i.e. $\rho < 1/2$), then only the efficient strategy A is played a significant fraction of time in the long run under small noise. Otherwise (i.e. $\rho > 1/2$), it is instead the inefficient strategy B which dominates the long-run behavior of the population.

As for Scenario I, if one is content with a definite, but less extreme criterion, of long-run predominance (i.e. the requirement of merely being the relatively most likely state), it is possible to find specific upper bounds on λ which ensure it. As a counterpart of (11) and (12), define

$$\tilde{\alpha}_\lambda \equiv \int_\rho^1 \phi_\lambda(\eta \mid 0)\,\mathrm{d}\eta \tag{14}$$

$$\tilde{\beta}_\lambda \equiv \int_{-1}^{-(1-\rho)} \phi_\lambda(\eta \mid 1)\,\mathrm{d}\eta \tag{15}$$

and note that, if $\rho \neq 1/2$ and A.2 holds,

$$(\tilde{\alpha}_\lambda - \tilde{\beta}_\lambda)(1/2 - \rho) > 0. \tag{16}$$

Recalling that $r \equiv 2^n - 1$, we consider the following two alternative assumptions:

A.3$'$ Let n and λ satisfy $\frac{1}{n^{2r}}\left(\frac{1-\tilde{\alpha}_\lambda}{1-\tilde{\beta}_\lambda}\right)^{r-n}\frac{\tilde{\alpha}_\lambda}{\tilde{\beta}_\lambda} > 1$.

A.3$''$ Let n and λ satisfy $\frac{1}{n^{2r}}\left(\frac{1-\tilde{\beta}_\lambda}{1-\tilde{\alpha}_\lambda}\right)^{r-n}\frac{\tilde{\beta}_\lambda}{\tilde{\alpha}_\lambda} > 1$.

Note that if $\rho < 1/2$ (resp. $\rho > 1/2$), then A.3$'$ (resp. A.3$''$) must hold for small enough λ. Thus, in this sense, each of these assumptions covers scenarios that are wider than those contemplated in (a) and (b) of Theorem 2, respectively.

Proposition 8 *Consider Scenario II.*
(a) If A.3$'$ *holds,* $\zeta^*(\mathbb{1}) > \zeta^*(\mu)$ *for all* $\mu \in \mathbf{supp}(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.
(b) If A.3$''$ *holds,* $\zeta^*(\mathbb{O}) > \zeta^*(\mu)$ *for all* $\mu \in \mathbf{supp}(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.

Corollary 4 *Consider Scenario II, and assume* A.2 *and* $n = 1$.
(a) If $\rho < 1/2$, $\zeta^*(\mathbb{1}) > \zeta^*(\mu)$, *for all* $\mu \in \mathbf{supp}(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.
(b) If $\rho > 1/2$, $\zeta^*(\mathbb{O}) > \zeta^*(\mu)$, *for all* $\mu \in \mathbf{supp}(\zeta^*) = \{0,1\}^n$, $\mu \neq \mathbb{1}$.

The conclusions established by the above results stand in marked contrast with those derived for Scenario I above (cf. Theorem 1, Proposition 5 and Corollary 2). In the next Section, I further elaborate on these differences and discuss their underlying intuition.

4 Discussion and Extensions

As explained, the *only* distinction between Scenarios I and II is based on an *ordinal* comparison of the off-equilibrium payoffs of the game, c and d.[19]

First, the inequality $d > c$ contemplated by Scenario I implies that, in any asymmetric encounter (i.e. one where the agents involved display a different strategy), the agent who adopts the efficient strategy performs better than her opponent. Such payoff advantage (which benefits the A agent only in *relative* terms) represents the essential driving force of Scenario I. Its effect is that, after every "uncoordinated" bilateral encounter, the A player is always promoted *if* the B player is. This fact becomes instrumental in breaking the symmetry between the only two configurations ($x = 0, 1$) that may be "robustly" sustained over time by concatenated generations.

Scenario II contemplates the opposite inequality $c > d$. In this case, the B player performs relative better than the A player in every asymmetric encounter. As a symmetric counterpart of the effect described above, this now has the effect of introducing a wedge in favor of strategy B, obtained at every uncoordinated encounter. However, since this strategy still has the handicap of enjoying a lower payoff at "equilibrium" situations (i.e. $b < a$), such positive bias only turns out to be fully effective in the long run when the selection rate prevailing in society is not too high ($\rho > 1/2$). Otherwise (i.e. when only less than half are promoted at each level), the fact that strategy A has the potential of attaining the highest possible payoff becomes the decisive factor in having the process select this strategy in the long run.

To further grasp the implications derived from Scenario II, it may be useful to resort to a paradigmatic application of it: the finitely-repeated Prisoner's Dilemma (PD), in a certain restricted version.[20] Suppose that every pair of players matched at each level play a long (but finite) repeated PD, action A being identified with the standard Tit-For-Tat strategy and B with continuous defection. Then, the payoff conditions of Scenario II are met, which leads to the conclusion that cooperative behavior will dominate in the long run only if the mechanism of social promotion is sufficiently selective.

Contrasting the conclusions derived in both scenarios with those obtained

[19]In this respect, it is interesting to observe that there are no specific conditions (beyond those defining each equilibrium) which concern how off- and on-equilibrium payoffs compare. In Scenario I, the inequality $d > c$ and the equilibrium conditions unambiguously determine the following *complete* ranking in payoffs: $a > b > d > c$. However, in Scenario II, the analysis is consistent with any comparison of c and b. In the language of Aumann (1993), this comparison pertains to whether the efficient equilibrium is self-enforcing ($b > c$) or not ($b < c$).

[20]I owe this illustration to Joe Harrington.

by existing evolutionary literature, the following two main differences arise:

1. Even though a definite criterion for long-run selection obtains,[21] the nature of it may crucially depend on "institutional" aspects of the environment. As one would expect, the selection of efficient behavior is favored by stringent selection conditions prevailing in the promotion mechanism of society.

2. The conditions on payoffs which determine the various possibilities are both of an ordinal nature and qualitatively different from the usual risk-dominance considerations so prevalent in received evolutionary literature. In particular, one has:

    (a) Efficient behavior is always selected, even when risk-dominated ($a + d < c + b$), if promotion is sufficiently selective ($\rho < 1/2$).

    (b) Inefficient behavior may also be selected in the long run despite being risk dominated ($a + d > c + b$), provided that it fares relatively better in asymmetric encounters ($d < c$) and promotion in society is not too selective ($\rho > 1/2$).

For the sake of focus, the theoretical framework postulated here is quite stylized in a number of different dimensions. To conclude, let me briefly discuss some possible variations and extensions of it which would enrich significantly the analysis.

Concerning the model's institutional features, both its promotion and the socialization components should be provided with a more flexible and realistic formulation. For example, it would be natural to allow for the option that agents who are not promoted at some given period may nevertheless remain eligible for promotion at a later time. On the other hand, it would also be interesting to accommodate for the possibility that agents who were promoted earlier in their life could be "demoted" later on if their performance were to fall below a certain level.

An alternative route of generalization pertains to the socialization component of the model. In the framework proposed, imitation by newcomers is exclusively focused on the uppermost level of society. Moreover, once those newcomers have adopted their decision, it is assumed to remain fixed for

[21] Here, I abstract from the considerations pertaining to the degree of sharpness with which the particular selection criterion operates, tailored to what is the approach pursued on limiting the magnitude of perturbation (i.e. whether we consider a "sufficiently small" λ as in Theorems 1 and 2, or rely simply on A.3, A.3$'$, or A.3$''$ as in Propositions 5 and 8).

the rest of their life. On both of these respects, it would be interesting to contemplate a less rigid formulation. First, one would like to allow for the possibility that intermediate levels of society might also have some potential for socializing (i.e. influencing) others at lower levels. Second, the ability of being socialized (or influenced) should not be taken to vanish abruptly once agents have lost their newcomer status.

Finally, other natural extensions of the model concern its payoff and interaction structure. A first possibility in this respect would involve the consideration of simple but asymmetric games played by agents belonging to distinct populations. An alternative approach would be to consider symmetric games (e.g. the familiar Hawk-Dove game) where the pure-strategy coordination is attained *via* asymmetric profiles. As explained above (cf. Footnote 4), the latter is one of the main tasks addressed in a companion paper.

Appendix

Proof of Proposition 2. As advanced, both of the conclusions contained in the Proposition follow from a general result on perturbed stochastic processes developed by Karandikar *et al.* (1996). For the sake of completeness, we next present a formal statement and proof of this result.

Let E be a compact state space and denote by $\Delta(E)$ its Borel probability measures. Given any set of transition probabilities $H \equiv \{H(\cdot \mid y)\}_{y \in E}$ and some probability measure $\nu \in \Delta(E)$, define the composition $\nu \cdot H$ as follows:

$$\nu \cdot H \equiv \int_E H(\cdot \mid y)\, \nu(\mathrm{d}y).$$

Similarly, any two transition probabilities H and G induce a third one $GH \equiv G \cdot H$ as follows:

$$GH(\cdot \mid y) = G(\cdot \mid y) \cdot H.$$

This allows us to define an m-step composition of H by:

$$H^m = H^{m-1} \cdot H$$

with the interpretation that, for all $y \in E$, $H^0(\cdot \mid y) = \delta_y$, i.e. the degenerate probability measure concentrated on y.

Now, given any $\varepsilon > 0$ and three transitions G, $\hat{G}$, and H, consider the perturbed transition H_ε defined, for each $y \in E$, as follows:

$$\begin{aligned} H_\varepsilon(\cdot \mid y) &= (1 - \psi(\varepsilon))\, H(\cdot \mid y) + \psi(\varepsilon)\, G_\varepsilon(\cdot \mid y) \\ G_\varepsilon(\cdot \mid y) &= (1 - \varphi(\varepsilon))\, G(\cdot \mid y) + \varphi(\varepsilon)\, \hat{G}(\cdot \mid y), \end{aligned}$$

where $\psi(\varepsilon) \to 0$, $\varphi(\varepsilon) \to 0$, $0 < \psi(\varepsilon) < 1$, $0 \leq \varphi(\varepsilon) < 1$, as $\varepsilon \to 0$.

The mentioned result reads as follows:

Theorem A (Karandikar *et al.* (1996)) *Consider G, H, and H_ε as above and assume that:*

(a) For each $y \in E$, the sequence $\{\frac{1}{m+1}\sum_{r=0}^{m} H^r(\cdot \mid y)\}_{m=0}^{\infty}$ converges weakly to some probability measure $W(\cdot \mid y) \in \Delta(E)$;

(b) G has the strong Feller property, i.e. for any real valued function f on E, $Gf(y) \equiv \int_E G(dz \mid y)f(z)$ is a continuous function of y for all bounded measurable f on E;

(c) G is open-set irreducible, i.e. for all open sets $U \subset E$ and every $y \in E$, $\sum_{r=0}^{\infty} G^r(y, U) > 0$; and

(d) GW has a unique invariant measure $\nu^* \in \Delta(\Omega)$.

Then, for all $\varepsilon > 0$, H_ε has a unique invariant measure ν_ε which converges weakly to ν^* as $\varepsilon \to 0$.

Proof. Given any $\varepsilon > 0$, properties (b) and (c) imply that H_ε is a T-chain. Applying Theorem 16.2.5 in Meyn and Tweedie (1993), H_ε is uniformly ergodic and has a unique invariant measure ν_ε. Then,

$$\nu_\varepsilon \cdot [(1 - \psi(\varepsilon))H + \psi(\varepsilon)G_\varepsilon] = \nu_\varepsilon$$

or

$$\psi(\varepsilon)\nu_\varepsilon \cdot G_\varepsilon = \nu_\varepsilon - (1 - \psi(\varepsilon))\, \nu_\varepsilon \cdot H.$$

Given any continuous (bounded) function f, apply the above probability measures to $H^r f$ as follows:

$$\psi(\varepsilon)\, \nu_\varepsilon \cdot G_\varepsilon\, H^r f = \nu_\varepsilon \cdot H^r f - (1 - \psi(\varepsilon))\, \nu_\varepsilon \cdot H\, H^r f \qquad (r = 0, ..., m).$$

Multiplying every one of the above respective expressions by $(1 - \psi(\varepsilon))^r$ and summing over $r = 0, ..., m$ one gets:

$$\psi(\varepsilon)\, \{(\nu_\varepsilon G_\varepsilon) \cdot \sum_{r=0}^{m} (1 - \psi(\varepsilon))^r H^r\} f = \nu_\varepsilon f - (1 - \psi(\varepsilon))^{m+1}\, \nu_\varepsilon H^{m+1} f.$$

Since $\sup_y |\nu_\varepsilon H^{r+1} f(y)| \leq \sup_y |f(y)|$, taking $m \to \infty$ in the previous expression leads to:

$$\nu_\varepsilon G_\varepsilon \varphi_\varepsilon = \nu_\varepsilon f \tag{17}$$

where

$$\varphi_\varepsilon(y) = \psi(\varepsilon) \sum_{r=0}^{\infty} (1 - \psi(\varepsilon))^r H^r f(y).$$

Since, from (a):

$$\lim_{m\to\infty}\frac{1}{m+1}\sum_{r=0}^{m} H^r f(y) = Wf(y),$$

we must also have, for all $y \in E$,

$$\lim_{\varepsilon\to 0}\varphi_\varepsilon(y) = Wf(y).$$

Moreover,

$$\nu_\varepsilon G_\varepsilon \varphi_\varepsilon = (1-\psi(\varepsilon))\nu_\varepsilon G\,\varphi_\varepsilon + \psi(\varepsilon)\nu_\varepsilon \hat{G}\,\varphi_\varepsilon.$$

Since $|\varphi_\varepsilon(y)| \leq \sup_{y'} |f(y)| \equiv M$, $\left|\nu_\varepsilon \hat{G}\,\varphi_\varepsilon\right| \leq M$. Thus, given that G is assumed strong Feller (and, therefore, both $G\varphi_\varepsilon$ and GWf are continuous) we have:

$$\sup_y |G\varphi_\varepsilon(y) - GWf(y)| \to 0$$

as $\varepsilon \to 0$. Hence, if $\{\varepsilon_n\} \to 0$ and $\nu_{\varepsilon_n} \to \hat{\nu}$ in the topology of weak convergence, then

$$\nu_{\varepsilon_n} G\varphi_{\varepsilon_n} \to \hat{\nu} GWf.$$

In view of (17), this implies:

$$\hat{\nu} f = \hat{\nu} GWf.$$

By (d), it follows that $\hat{\nu} = \nu^*$, as desired.

Now, with the purpose of applying Theorem A to our context, identify $E = \Omega$ and make $H(\cdot \mid \mu) = \delta_{F(\mu)}$ for each $\mu \in \Omega$ where, as before, $\delta_{F(\mu)}$ stands for the measure fully concentrated in $F(\mu)$. Then, we have $W(\cdot \mid \mu) = R(\cdot \mid \mu)$ for each μ, where the latter is defined as the weak limit of (6). Furthermore, make $G = Q$ and $\hat{G} = \hat{Q}$ where $\hat{Q}$ stands for the Markov process embodying the transitions which can be achieved in the model relying on at least two simultaneous perturbations at different levels.

Next define

$$\begin{aligned}
\psi(\varepsilon) &= 1-(1-\varepsilon)^n \\
\varphi(\varepsilon) &= 1-\frac{n\varepsilon(1-\varepsilon)^{n-1}}{1-(1-\varepsilon)^n}.
\end{aligned}$$

On the basis of these specifications, the perturbed process P_ε may be formulated as follows:

$$\begin{aligned}
P_\varepsilon(\cdot \mid \mu) &= (1-\psi(\varepsilon))\,\delta_{F(\mu)} + \psi(\varepsilon)\,Q_\varepsilon(\cdot \mid \mu) \\
Q_\varepsilon(\cdot \mid \mu) &= (1-\varphi(\varepsilon))Q(\cdot \mid \mu) + \varphi(\varepsilon)\,\hat{Q}(\cdot \mid \mu)
\end{aligned}$$

which finally permits identifying P_ε with the process H_ε contemplated in Theorem A.

Conditions (b) and (c) of this theorem are satisfied. Concerning (a), this is established below by Propositions 4 and 7, for Scenarios I and II respectively. Thus, in order to apply Theorem A, it only remains to show that its condition (d) applies.

To this effect, it is enough to confirm that there is some given state $\hat{\mu}$ such that, from *every* other state $\mu \in \Omega$, $(QR)^m(\hat{\mu} \mid \mu) > 0$ for some finite m. For concreteness, we choose $\hat{\mu} = \mathbb{1} = (1, 1, ..., 1)$ and restrict the discussion to Scenario I. The argument for Scenario II is fully analogous.

Choose any $\mu \in \Omega$, $\mu \neq \mathbb{1}$, and denote by $\chi(\mu) < n$ the number of components in μ equal to 1. Let

$$\ell(\mu) \equiv \max\{\chi(\mu') : \mu' \in \mathbf{supp}\,[QR(\cdot \mid \mu)]\}. \tag{18}$$

If $\ell(\mu) = n$, we are done. Thus, suppose $\ell(\mu) = r < n$. By Proposition 4, there must be some $\mu^1 \in \mathbf{supp}[QR(\cdot \mid \mu)]$ such that $\ell(\mu^1) = r$ and $\mu^1_n \in \{0, \xi\}$. Suppose, for concreteness, that $\mu^1_n = 0$.

Denote by $\hat{\mu}^1 = (\hat{\mu}^1_1, \hat{\mu}^1_2, ..., \hat{\mu}^1_n)$ the state obtained from μ^1 after a perturbation of magnitude η has impinged on the socialization dynamics and there has been *one* iteration of the promotion dynamics (i.e., an iteration of the function $F(\cdot)$). That is:

$$\begin{aligned} \hat{\mu}^1_1 &= f(\mu^1_n + \eta) = f(\eta) \\ \hat{\mu}^1_k &= f(\hat{\mu}^1_{k-1}) \qquad (k = 2, ..., n). \end{aligned}$$

Mere inspection of (7) indicates that:

$$x \geq \sqrt{\rho} \Rightarrow f(x) = 1.$$

Furthermore, recall that Proposition 3 establishes that $f(\xi) = \xi$. Hence, there is probability no smaller than $\int_{\sqrt{\rho}}^{1} \phi_\lambda(\eta)\, d\eta > 0$ that $\chi(\hat{\mu}^1) = r + 1$. Therefore,[22]

$$\mu^2 \in \mathbf{supp}\,[R(\cdot \mid \hat{\mu}^1)] \Rightarrow \chi(\mu^2) = r + 1.$$

In analogy with the notational convention used in (18), this means that

$$\ell^2(\mu) \equiv \max\{\chi(\mu') : \mu' \in \mathbf{supp}\,[(QR)^2(\cdot \mid \mu)]\} = r + 1.$$

Clearly, this line of argument may proceed until $\ell^n(\mu) = n$, which then completes the proof of Proposition 2. ■

[22]Proposition 4 implies that, given any $\mu \in \Omega$ and every $\mu', \mu'' \in \mathbf{supp}[QR(\cdot \mid \mu)]$, one has $\chi(\mu') = \chi(\mu'')$.

Proof of Proposition 3. First, we observe that there must exist at least one interior fixed point since $f(\cdot)$ is continuous and, from (a) and (f) in (7), it follows that there are some $x' < x''$ such that $f(x') < x'$ and $f(x'') > x''$. Thus, all conclusions of the Proposition are established if it is shown that there is at most one interior fixed point of $f(\cdot)$.

To this effect, it is useful to consider two cases. First, when $\rho \leq \frac{1}{2}$, ranges (b)-(d) collapse and $f(x) = \frac{1}{\rho}x^2$ for all $x \leq \sqrt{\rho}$. The unique interior fixed point is then $x^* = \rho \, (= \xi)$, since $f(x) = 1$ for $x \geq \sqrt{\rho}$.

Consider now the complementary case $\rho > \frac{1}{2}$. Then, $x = \rho$ does not belong to either range (a) or (e), where $f(x) = \frac{1}{\rho}x^2$. Thus any interior fixed point of $f(\cdot)$ must be sought in ranges (b)-(e). Obviously, the linear function applicable in (c) can be ruled out. Therefore, we are left with ranges (b) and (d), where $f(x) = \frac{1}{\rho}(2x - x^2 + \rho - 1)$. It can be checked that the unique interior solution to

$$\frac{1}{\rho}(2x - x^2 + \rho - 1) = x$$

is $x^* = 1 - \rho$, which completes the proof. ■

Proof of Proposition 4. To facilitate matters, it is convenient to reformulate the dynamic system induced by $F(\cdot)$ – cf. (1) and (2) – as follows. Let $[\cdot]_n$ stand for the "modulo n" operator. Given any level $k = 1, 2, ..., n$ and any $t \in \mathbb{N}$, define $\iota(k,t)$ as the (unique) index $j = 1, 2, ..., n$ such that $[j+t]_n = [k]_n$. Relying on this notational convention, the state paths $\{\mu(t)\}_{t=0}^{\infty}$ induced by the unperturbed dynamical system are of the following form:

$$\mu_k(t) = f^t(\mu_{\iota(k,t)}(0)) \qquad (k = 1, 2, ..., n; \quad t = 1, 2, ...) \tag{19}$$

Suppose first that $\mu_k(0) \in \{0, 1, \xi\}$ for each $k = 1, 2, ..., n$. Then, by Proposition 3, it follows that $[\mu(0), \mu(1), ..., \mu(n-1)]$ is an n-cycle in Λ_I.

Suppose instead that $\mu_k(0) \notin \{0, 1, \xi\}$ for some k. Then, partition the different levels $k = 1, 2, ..., n$ into three sets:

$$\begin{aligned} A &= \{k : \mu_k(0) > \xi\} \\ B &= \{k : \mu_k(0) < \xi\} \\ C &= \{k : \mu_k(0) = \xi\}. \end{aligned}$$

Choose any arbitrary $\eta > 0$. In view of (8) and (19), there exists some large enough T such that if $t \geq T$:

$$\mu_k(t) \begin{cases} \geq 1 - \eta & \text{if } \iota(k,t) \in A \\ \leq \eta & \text{if } \iota(k,t) \in B \\ = \xi & \text{if } \iota(k,t) \in C. \end{cases}$$

Thus, since $\iota(k,t) = \iota(k,t+rn)$ for all $k = 1,2,..,n$ and every $r \in \mathbb{N}$, (19) implies that:

$$\lim_{r\to\infty} \mu_k(t+rn) \begin{cases} =1 & \text{if } \iota(k,t) \in A \\ =0 & \text{if } \iota(k,t) \in B \\ =\xi & \text{if } \iota(k,t) \in C. \end{cases}$$

Suppose that t has been chosen of the form $t = r_0 n + 1$ for some $r_0 \in \mathbb{N}$. Then,

$$\lim_{r\to\infty} [\mu(rn+1), \mu(rn+2), ..., \mu(rn+n)] = (\hat{\mu}^1, \hat{\mu}^2, ..., \hat{\mu}^n) \in \Lambda_I \,,$$

where

$$\hat{\mu}_k^q = F(\hat{\mu}_k^{q-1}) \qquad (q = 2, ..., n),$$

and $\hat{\mu}_k^1 = 1, 0, \xi$ if, respectively, $\iota(k,t) \in A, B, C$. This completes the proof of the Proposition. ■

Proof of Theorem 1. By Proposition 2, the distribution ζ^* is the unique invariant measure of the Markov process with transition $QR(\cdot \mid \mu)$. By Proposition 4, this process can be conceived as Markov chain on the *finite* state space $\Gamma \equiv \{0, 1, \xi\}^n \subset \Omega$.

Hence, to characterize ζ^* for any given $\lambda > 0$ we may rely on the graph-theoretic methods developed by Freidlin & Wentzel (1984) and frequently used by recent Evolutionary Theory. Particularized to our framework, they involve the following key concept.

Definition Let $\mu \in \Gamma$. A *μ-tree* Y is a directed graph on Γ (i.e. a subset of $\Gamma \times \Gamma$) such that every state $\mu' \in \Gamma \backslash \{\mu\}$ is the initial point of exactly one "arrow" $(\mu', \mu'') \in Y$ and from any such state μ' there is a *path* $\{(\mu', \mu''), (\mu'', \mu'''), ..., (\mu^{(r-1)}, \mu^{(r)})\} \subset Y$ whose end point $\mu^{(r)} = \mu$.

For every pair $\mu, \mu' \in \Gamma$, denote $p(\mu, \mu') \equiv QR(\mu' \mid \mu)$. To reflect the dependence of these transition probabilities on λ, we shall find it useful to write them as $p_\lambda(\mu, \mu')$.

For any given $\mu \in \Gamma$, let $\mathcal{Y}_\mu$ stand for the (finite) set of all μ-trees. We define the vector $q_\lambda(\mu) \equiv (q_\lambda(\mu))_{\mu\in\Gamma}$ as follows:

$$q_\lambda(\mu) \equiv \sum_{Y\in\mathcal{Y}_\mu} \prod_{(\mu',\mu'')\in Y} p_\lambda(\mu', \mu''). \tag{20}$$

The following Theorem is an application of a result by Freidlin & Wentzel (1984, p. 177) to our context.

Theorem B *The invariant distribution* ζ^* *is given by:*

$$\zeta^*(\mu) = \frac{1}{\sum_{\mu' \in \Gamma} q_\lambda(\mu')} q_\lambda(\mu) \qquad (\mu \in \Gamma).$$

The remaining part of the proof involves finding appropriate upper or lower bounds for each $q_\lambda(\mu)$, $\mu \in \Omega$, as a function of λ. First, note that the state space Γ can be partitioned into some q equivalence classes $\{\mathcal{A}_i\}_{i=1}^q$, each one of them consisting of all those states belonging to the same n-cycle for the unperturbed dynamics. That is, for any such $\mathcal{A}_i$,

$$\mu, \mu' \in \mathcal{A}_i \Rightarrow \mu' = F^r(\mu),\ r \leq n.$$

Consider now a $\mathbb{1}$-tree Y with the following two characteristics:

(i) $\forall \mathcal{A}_i \neq \{\mathbb{1}\}$ there exists a *unique* $\hat{\mu} \in \mathcal{A}_i$ such that $[\mu \in \mathcal{A}_i,\ \mu \neq \hat{\mu},\ (\mu, \mu') \in Y] \Rightarrow \mu' \in \mathcal{A}_i$,

(ii) $\forall \mathcal{A}_i \neq \{\mathbb{1}\}$, $[\mu \in \mathcal{A}_i,\ \mu' \notin \mathcal{A}_i,\ (\mu, \mu') \in Y] \Rightarrow \chi(\mu') = \chi(\mu) + 1$,

where recall that $\chi(\mu)$ stands for the number of components in μ equal to 1 (cf. proof of Proposition 2). Some such $\mathbb{1}$-tree Y can be readily constructed.

Denote $|\Gamma| = 3^n \equiv \gamma$ and recall the definitions of α_λ and β_λ found in (11) and (12). Since it is assumed that $\rho \neq 1/2$ (and thus $\xi < 1/2$), Assumption A.2 implies that $\alpha_\lambda > \beta_\lambda$ for all $\lambda > 0$. Now, for any $\mathbb{1}$-tree Y satisfying (i) and (ii) above, consider the following lower bound on its induced product of transition probabilities:

$$\prod_{(\mu', \mu'') \in Y} p_\lambda(\mu', \mu'') \geq \frac{1}{n}(1 - \alpha_\lambda)^\gamma \, \alpha_\lambda^{q-1},$$

Here, we rely on the fact that for transitions across states belonging to the same $\mathcal{A}_i$, all states have the same prior probability of being reached according to $R(\cdot \mid \mu)$ (in particular, no smaller than $\frac{1}{n}$). Obviously, by (20), the same lower bound applies to $q_\lambda(\mathbb{1})$, that is:

$$q_\lambda(\mathbb{1}) \geq \frac{1}{n}(1 - \alpha_\lambda)^\gamma \, \alpha_\lambda^{q-1}. \tag{21}$$

Consider now any μ-tree for $\mu \in \Gamma$, $\mu \neq \mathbb{1}$. This tree must have a path y linking state $\mathbb{1}$ to μ. Since $\chi(\mu) < 1 = \chi(\mathbb{1})$, one of the arrows in this tree (μ', μ'') must involve $\chi(\mu') > \chi(\mu)$, its associated transition probability being no larger that β_λ. This implies that:

$$q_\lambda(\mu) \leq K \, \beta_\lambda \alpha_\lambda^{q-2}, \tag{22}$$

for some K which can be chosen equal to the largest cardinality $|\mathcal{Y}_\mu|$ across all $\mu \in \Gamma$.

Choose any $\delta > 0$, as contemplated in the statement of the Theorem. Then, make

$$\vartheta = \delta \left(\frac{1}{2}\right)^{\gamma} \frac{1}{\gamma\, n\, K}. \tag{23}$$

Given such ϑ, one may invoke Assumption A.1 to assert that there exist some $\bar{\lambda}$ such that if $0 < \lambda \leq \bar{\lambda}$,

$$\frac{\beta_\lambda}{\alpha_\lambda} \leq \vartheta, \quad \alpha_\lambda \leq \frac{1}{2}.$$

In view of (21), (22), and (23), this implies that, for any such λ:

$$\frac{q_\lambda(\mu)}{q_\lambda(\mathbb{1})} \leq \frac{1}{\gamma}\delta$$

for any $\mu \in \Gamma$, $\mu \neq \mathbb{1}$. Therefore,

$$q_\lambda(\mathbb{1}) \geq 1 - \sum_{\mu \neq \mathbf{1}} q_\lambda(\mu) \geq 1 - \frac{\gamma - 1}{\gamma}\delta \geq 1 - \delta,$$

which is the desired conclusion, thus completing the proof. ■

Proof of Proposition 5. Denote by $\Gamma_0 \equiv \{0,1\}^n \subset \Gamma$. In view of Proposition 4 and its Corollary 1, all those states $\mu \notin \Gamma_0$ are transitory and, therefore, the support of ζ^* must be included in Γ_0. Clearly, this allows us to restrict the graph-theoretic techniques of Freidlin and Wentzel (1984) described above to apply only to states in Γ_0.

With this simplifying restriction in place, Theorem B, and Assumption A.3, the desired conclusion is a direct consequence of the following Claim.

Claim: Consider any $\hat{\mu} \in \Gamma_0$, $\hat{\mu} \neq \mathbb{1}$.There is a one-to-one mapping Υ between the set of $\hat{\mu}$-trees $\mathcal{Y}_{\hat{\mu}}$ and the set of $\mathbb{1}$-trees $\mathcal{Y}_{\mathbf{1}}$ (restricted to Γ) such that, for each $\hat{\mu}$-tree $\hat{Y}$, the corresponding $\mathbb{1}$-tree $\tilde{Y} = \Upsilon(\hat{Y})$ satisfies

$$\prod_{(\mu',\mu'')\in\hat{Y}} p_\lambda(\mu',\mu'') < \prod_{(\mu',\mu'')\in\tilde{Y}} p_\lambda(\mu',\mu''). \tag{24}$$

To establish this Claim, choose some $\hat{\mu} \in \Gamma_0$ and any associated $\hat{\mu}$-tree $\hat{Y}$. Let $\hat{y} \subset \hat{Y}$ be the path joining the state $\mathbb{1}$ to $\hat{\mu}$. To construct the associated c-tree $\tilde{Y} = \Upsilon(\hat{Y})$, first reverse every arrow $(\mu',\mu'') \in$ to the corresponding one (μ'', μ'). Let $\tilde{y}$ denote the associated path joining $\hat{\mu}$ to state $\mathbb{1}$. Then, we simply make $\tilde{Y} = (\hat{Y} \backslash \hat{y}) \cup \tilde{y}$. This procedure leads a well-defined $\mathbb{1}$-tree. On the other hand, it defines a one-to-one mapping since it is injective and onto. (Simply note that the converse reversal procedure associates to each $\mathbb{1}$-tree a unique $\hat{\mu}$-tree.)

Next, (24) is verified. By construction, one has

$$\frac{\prod_{(\mu',\mu'')\in\tilde{Y}} p_\lambda(\mu',\mu'')}{\prod_{(\mu',\mu'')\in\hat{Y}} p_\lambda(\mu',\mu'')} = \frac{\prod_{(\mu',\mu'')\in\tilde{y}} p_\lambda(\mu',\mu'')}{\prod_{(\mu',\mu'')\in\hat{y}} p_\lambda(\mu',\mu'')} \equiv \sigma \tag{25}$$

Thus, to complete the proof, it is enough to show that, under Assumption A.3, $\sigma > 1$. To this effect, it is useful to specify a list of bounds on the variation of transition probabilities across any two states $\mu', \mu'' \in \Gamma_0$ with $|\chi(\mu') - \chi(\mu'')| \leq 1$ (recall that $\chi(\mu)$ denotes the number of unit components of μ):[23]

(1) $\chi(\mu') = \chi(\mu'') \Rightarrow \frac{1}{n^2}(1-\alpha_\lambda) \leq p_\lambda(\mu',\mu'') \leq 1 - \beta_\lambda$.

(2) $\chi(\mu'') > \chi(\mu') \Rightarrow \frac{1}{n^2}\alpha_\lambda \leq p_\lambda(\mu',\mu'') \leq \alpha_\lambda$.

(3) $\chi(\mu'') < \chi(\mu') \Rightarrow \frac{1}{n^2}\beta_\lambda \leq p_\lambda(\mu',\mu'') \leq \beta_\lambda$.

These inequalities readily follow from the specification of the process QR and, in particular, the fact that perturbations at each level are postulated equi-probable.

Consider now the $\tilde{y}$ path and denote by $\tilde{q}_1$, $\tilde{q}_2$ and $\tilde{q}_3$ the number of transitions $(\mu', \mu'') \in \tilde{y}$ for which, respectively, (1), (2), or (3) above apply. Since $\chi(\mathbb{1}) = n$, we have:

$$n - \chi(\hat{\mu}) = \tilde{q}_2 - \tilde{q}_3 > 0.$$

Similarly, if $\hat{q}_1$, $\hat{q}_2$ and $\hat{q}_3$ denote the number of transitions $(\mu', \mu'') \in \hat{y}$ for which, respectively, (1), (2), or (3) above apply, one has $\hat{q}_1 = \tilde{q}_1$, $\hat{q}_2 = \tilde{q}_3$, and $\hat{q}_3 = \tilde{q}_2$. From these considerations and the bounds specified in (1)-(3), it follows that

$$\begin{aligned}
\prod_{(\mu',\mu'')\in\tilde{y}} p_\lambda(\mu',\mu'') &\geq \left(\frac{1}{n^2}(1-\alpha_\lambda)\right)^{\tilde{q}_1} \left(\frac{1}{n^2}\alpha_\lambda\right)^{\tilde{q}_2} \left(\frac{1}{n^2}\beta_\lambda\right)^{\tilde{q}_3} \\
\prod_{(\mu',\mu'')\in\hat{y}} p_\lambda(\mu',\mu'') &\leq (1-\beta_\lambda)^{\hat{q}_1} \, (\alpha_\lambda)^{\hat{q}_2} \, (\beta_\lambda)^{\hat{q}_3} \\
&= (1-\beta_\lambda)^{\tilde{q}_1} \, (\beta_\lambda)^{\tilde{q}_2} \, (\alpha_\lambda)^{\tilde{q}_3}
\end{aligned}$$

[23]Note the only transitions (μ', μ'') that have positive probability according to QR are those for which $|\chi(\mu') - \chi(\mu'')| \leq 1$.

which, in view of (25), implies:

$$\sigma \geq \left(\frac{1}{n^2}\right)^{\tilde{q}_1+\tilde{q}_2+\tilde{q}_3} \left(\frac{1-\alpha_\lambda}{1-\beta_\lambda}\right)^{\tilde{q}_1} \left(\frac{\alpha_\lambda}{\beta_\lambda}\right)^{\tilde{q}_2-\tilde{q}_3}.$$

Since $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 \leq r$, $\tilde{q}_1 \leq r - n$, and $\tilde{q}_2 - \tilde{q}_3 \geq 1$, Assumption A.3 implies that $\sigma > 1$, as desired, thus completing the proof. ■

Proof of Proposition 6. First, observe that at least one interior fixed point must exist since the function $f(\cdot)$ specified in (13) is continuous and, from (a') and (d'), it follows that there are some $x' < x''$ such that $f(x') < x'$ and $f(x'') > x''$. It is immediate to see that such fixed points can only exist in ranges (a') and (c'), in both of which we have $f(x) = \frac{1}{\rho}x^2$. The only interior fixed point of this function is $x^* = \rho$, which completes the proof. ■

Proof of Proposition 7. The proof is exactly as that of Proposition 4 with ρ substituting ξ. ■

Proof of Theorem 2. Recall the definitions of $\tilde{\alpha}_\lambda$ and $\tilde{\beta}_\lambda$ introduced in (14) and (15). If $\rho < 1/2$, then $\tilde{\alpha}_\lambda = \alpha_\lambda$ and $\tilde{\beta}_\lambda = \beta_\lambda$, where α_λ and β_λ are defined in (11) and (12), respectively. Then, the same argument used in the proof of Theorem 1 can be applied to derive the desired conclusion.

Consider the alternative case where $\rho > 1/2$. Assumption A.2 implies that $\tilde{\alpha}_\lambda < \tilde{\beta}_\lambda$. A line of argument analogous to that used for Theorem 1 (but now applied to state $\mathbb{O}$ rather than $\mathbb{1}$) leads to the following inequalities:

$$q_\lambda(\mathbb{O}) \geq \frac{1}{n}(1 - \tilde{\beta}_\lambda)\, \tilde{\beta}_\lambda^{q-1}$$

and, for all $\mu \in \{0, 1, \rho\}^n$, $\mu \neq \mathbb{O}$,

$$q_\lambda(\mu) \leq K\, \tilde{\alpha}_\lambda\, \tilde{\beta}_\lambda^{q-2}$$

for some large enough K.

Thus, given any $\delta > 0$, choose ϑ as in (23) and select $\bar{\lambda}$ small enough such that if $0 < \lambda \leq \bar{\lambda}$, then:

$$\frac{\tilde{\alpha}_\lambda}{\tilde{\beta}_\lambda} \leq \vartheta, \quad \tilde{\beta}_\lambda \leq \frac{1}{2}.$$

This implies:

$$\frac{q_\lambda(\mu)}{q_\lambda(\mathbb{O})} \leq \frac{1}{\gamma}\delta$$

and therefore $q_\lambda(\mathbb{O}) \geq 1 - \delta$, as desired. This completes the proof of the Theorem. ■

Proof of Proposition 8. First, suppose $\rho < 1/2$ which, as indicated in the proof of Theorem 2, implies that

$$\tilde{\alpha}_\lambda = \alpha_\lambda > \beta_\lambda = \tilde{\beta}_\lambda$$

where α_λ and β_λ are given by (11) and (12), and $\tilde{\alpha}_\lambda$ and $\tilde{\beta}_\lambda$ are defined in (14) and (15).Then, the argument of Proposition 5 can be replicated without change.

If instead $\rho > 1/2$, then $\tilde{\alpha}_\lambda < \tilde{\beta}_\lambda$ and a proof analogous to that of Proposition 5 can be applied, now substituting $\tilde{\alpha}_\lambda$ for β_λ, $\tilde{\beta}_\lambda$ for α_λ, and the frequency $x = 0$ for $x = 1$ in the components of states $\mu \in \Gamma_0$. Any details should be apparent, being a straightforward counterpart of those found in the former proof. ■

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